Q: Given a vector subspace $V \subseteq \mathbb{R}^m$ with a basis

$S = \{u_1, u_2, \cdots, u_n\}$ and $\tilde{S} = \{v_1, v_2, \cdots, v_k\}$.

What is the relationship between S and $\tilde{S}$??

Thm: We have $n = k$.

i.e. Any two basis of V have the same numbers of elements.

Rmk: Then we call such number, the dimension of V, denoted by $\dim(V)$.

Lemma: If $u_1, u_2, \cdots, u_m$ are linearly indep. in $V \subseteq \mathbb{R}^k$

and V has the basis $v_1, v_2, \cdots, v_n$,

then $m \leq n$.

pf: Same as the case when $V = \mathbb{R}^n = \operatorname{span}\{e_k \mid k = 1, 2, \cdots, n\}$

(only a sketch): Suppose $m > n$.

$\because u_i \in \operatorname{span}\{v_1, \cdots, v_n\}$

$\therefore\ u_i = \sum_{j=1}^{n} \lambda_{ji} v_j$ (uniquely)

$$\Rightarrow \underbrace{[v_1 \ \cdots \ v_n]}_{B} \underbrace{\begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1m} \\ \lambda_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nm} \end{bmatrix}}_{C} = [u_1 \cdots u_m]$$

Linearly indep $\Rightarrow$ There are no $x \neq 0 \in \mathbb{R}^m$ s.t.

$(BCx \Rightarrow) Ax = 0$

$\because C = n \times m$ matrix with $m > n$.

$\therefore \text{Null}(C) \supsetneqq \{0\}$ i.e. $\exists y \in \mathbb{R}^m$ s.t. $Cy = 0$

(think about RREF of C if you forget)

$\Rightarrow BCy = Ay = 0 \to \leftarrow$. $\therefore m \leq n$.

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pf of thm: If $S, \tilde{S}$ are basis of V

$S = \{u_1, \cdots, u_m\}$ is basis of $V = \text{span}(S)$
$= \text{span}\{v_1, \cdots, v_k\}$

$\Rightarrow m \leq k$

Interchange S and $\tilde{S}$ $\Rightarrow m \geq k \Rightarrow m = k$ #.

Extension thm:

Thm Let W be non-zero subspace of $\mathbb{R}^n$.

If $v_1, v_2, \dots, v_k$ are linearly indep. in W

then $\exists\ v_{k+1}, v_{k+2}, \dots, v_r$ $(r = \dim W)$,

s.t. $v_1, \dots, v_r$ is basis of W.

pf: If $k = r$, we claim that $v_1, \dots, v_r$ is a basis of W. Suffices to show that

$$\forall x \in W,\ x \in \operatorname{span}\{v_1, \dots, v_r\}.$$

If not, then $\{v_1, \dots, v_r, x\}$ is linearly indep

$$\Rightarrow\ r+1 \le \dim(W) = r \rightarrow\leftarrow.$$

$\therefore\ x \in \operatorname{span}\{v_1, \dots, v_r\}$.

If $k < r$, $\exists\ v_{k+1} \in W$ s.t. $\overset{\neq 0}{v_{k+1}} \notin \operatorname{span}\{v_1, \dots, v_k\}$

$$\Rightarrow\ \{v_1, v_2, \dots, v_{k+1}\} \text{ is linearly indep.}$$

If $k+1 = r$, done. Otherwise, find v_{k+2} s.t.

$$\{v_1, v_2, \dots, v_k, v_{k+1}, v_{k+2}\} \text{ is linearly indep.}$$

The process stop at the l-th step where

$k + l = r$. #

Next que: Given a set $S = \{u_1, \cdots, u_n\} \subseteq \mathbb{R}^m$.

want to find some simple basis of $\mathrm{span}(S)$.

Eg: $u_1 = \begin{bmatrix} 7 \\ 6 \\ 12 \\ 33 \end{bmatrix}, u_2 = \begin{bmatrix} 5 \\ 5 \\ 7 \\ 24 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix}$

then $\mathrm{span}\{u_1, u_2, u_3\} = \mathrm{span}\{v_1, v_2, v_3\} = V$

where $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ $\leftarrow$ basis for V.

Q: How to find them systematically??

Row operation: preserve the structure of span.

But simplifying the rows. ...

Using this to find basis.

Defn: Given a $m \times n$ matrix A.

The row space of A, $R(G)$, is defined to be $C(A^t)$ where A^t = transpose of A.

(A^t : $n \times m$ matrix given by $(A^t)_{ij} = A_{ji}$)

Eg: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$

Observe $\overset{(m \times n)}{A} = \begin{bmatrix} \text{—} v_1 \text{—} \\ \text{—} v_2 \text{—} \\ \vdots \\ \text{—} v_m \text{—} \end{bmatrix}$ $m \times n$ matrix (where $v_i^t \in \mathbb{R}^n$ as column vector)

then $R(A) = \text{span}\{ v_1^t, v_2^t, \ldots, v_m^t \} \subseteq \mathbb{R}^n$ (as subspace)

If G = $m \times m$ matrix

then $GA = \begin{bmatrix} \text{—} u_1 \text{—} \\ \vdots \\ \text{—} u_m \text{—} \end{bmatrix}$, $G = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ G_{21} & \ddots & \\ \vdots & & \ddots \\ G_{m1} & & G_{mm} \end{bmatrix}$

Then $\begin{cases} u_1 = G_{11}v_1 + G_{12}v_2 + \cdots + G_{1m}v_m \\ \vdots \\ u_m = G_{m1}v_1 + G_{m2}v_2 + \cdots + G_{mm}v_m. \end{cases}$

$\therefore$ each $u_i^t \in \operatorname{span}\{v_1^t, \cdots, v_m^t\}$

$$\therefore \quad R(GA) = \operatorname{span}\{u_1^t, \cdots, u_m^t\}$$
$$\subseteq \operatorname{span}\{v_1^t, \cdots, v_m^t\} = R(A).$$

If G = non-singular,

applying above argument with $G \to G^{-1}$, $A \to GA$ ..

$$\Rightarrow \quad R(A) = R(G^{-1} \cdot GA) \subseteq R(GA)$$

$\Downarrow$

<u>Thm</u>: Suppose $G = m\times m$ matrix and $A = m\times n$ matrix

then $R(GA) \subseteq R(A)$.

Moreover if G = non-singular, then $R(GA) = R(A)$.

<u>Thm</u>: Suppose $A = m\times n$ matrix and $A' =$ RREF of A.

Then ① $R(A) = R(A')$

② If $\operatorname{rank}(A) \left(= \operatorname{rank}(A')\right) = r > 0$, and $A' = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_r' \\ v_{r+1}' \\ \vdots \\ v_m' \end{bmatrix}$, then $(v_1')^t, \cdots, (v_r')^t$ is a basis for $R(G)$.

<u>Example</u>: $u_1 = \begin{bmatrix} 7 \\ 6 \\ 12 \\ 33 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ 5 \\ 7 \\ 24 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 5 \end{bmatrix}$

want to find a basis for $V = \text{span}\{u_1, u_2, u_3\}$.

i.e. want to find basis for $C(A)$ where

$$A = \begin{bmatrix} 7 & 5 & 1 \\ 6 & 5 & 0 \\ 12 & 7 & 4 \\ 33 & 24 & 5 \end{bmatrix} \qquad C(A) = R(A^t)$$

$$G = A^t = \begin{bmatrix} 7 & 6 & 12 & 33 \\ 5 & 5 & 7 & 24 \\ 1 & 0 & 4 & 5 \end{bmatrix} \longrightarrow G' = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$\therefore \text{rank}(G') = 3$

$\therefore (v_1')^t = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}$, $(v_2')^t = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}$, $(v_3')^t = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for V.

Eg 2: $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \\ -1 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -5 \\ 2 \\ 0 \end{bmatrix} \right\}$, $V = \text{span}(S)$.

find a basis for V

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & -1 \\ 7 & 3 & 5 & -5 \\ 1 & 1 & -1 & 2 \\ -1 & 0 & 9 & 0 \end{bmatrix}$$ find basis for $C(A)$.

$$G = A^t = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$$

$$\rightarrow G' = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ $rk(G') = 3$.

basis $= \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ for V.

Why works??

Finding basis : considering all possible linear combination

row operations : special case of row operation!!